# REGULARIZING THE PROBLEM OF GAMES ENCOUNTER OF MOTIONS

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The problem [1–6] of the control $u$ which ensures convergence of the pursuing motion $y\,[t]$ with the pursued motion $z\,[t]$ is considered. The regularization of the extremal aiming rule [6 and 7] announced in paper [8] is investigated.

**1.** Let us consider the pursuing ($y\,[t]$) and pursued ($z\,[t]$) motions described by the differential equations

$$dy/dt = f^{(1)}[y,\ u] \tag{1.1}$$

$$dz/dt = f^{(2)}[z,\ v] \tag{1.2}$$

Here $y$, $z$ are the $n$-dimensional phase vectors of the objects; $u$, $v$ are the $r$-dimensional control vectors; $f^{(i)}$ are known differentiable vector functions. The process begins at some instant $t = t_0$. The realizations $u\,[t]$ and $v\,[t]$ of the controls $u$ and $v$ are restricted by the conditions

$$u\,[t] \in U, \qquad v\,[t] \in V \qquad (t \geqslant t_0) \tag{1.3}$$

where $U$, $V$ are given bounded domains in the spaces $\{u\}$ and $\{v\}$. Encounter at the instant $t = t_*$ is defined as the situation satisfying the condition

$$\{y\,[t_*]\}_m = \{z\,[t_*]\}_m \tag{1.4}$$

(The number $m$ is given; the symbol $\{w\}_m$ denotes the vector consisting of the first $m$ components of the vector $w$.) The pursuer's objective is to achieve encounter. The control $u$ is formed by the feedback principle at each instant $t \geqslant t_0$ on the basis of information concerning the situation until this instant. We assume that the pursuer can meet with any permissible realization $v\,[t]$ of the control. Construction of a workable control in the form of the function

$$u\,[t] = u\,(y\,[t],\ z\,[t]) \tag{1.5}$$

involves certain difficulties [1–6]. One way to overcome these difficulties is to enlarge the set of arguments in (1.5). In this paper we shall investigate one such method of regularizing the problem. This method is based on the extremal aiming rule [6], which is in turn based on the retention of the attainability domain $G^{(2)}\,[z\,[t],\ \eta\,[t]]$ of the motion $z\,[t]$ in the attainability domain $G^{(1)}\,[y\,[t],\ \eta\,[t]]$ of the motion $y\,[t]$ during variation of the time $t$ under the condition that the absorption time $\vartheta^\circ\,[t] = t + \eta\,[t]$ does not increase (e.g. see [7], p. 331). We shall investigate the regularization of this rule which adds the argument $\eta\,[t]$ to (1.5) and introduces a certain time lag $\xi$ into the aiming condition. This method of regularization is proposed in [8]. It is realized in the form of a discrete approximating scheme which differs from the regularizing schemes described in [8–10] in that it also introduces the time lag $\xi$. This broadens the opportunities for control, but gives rise to new problems about the stability of the process. Investigation of this stability exceeds the scope of the present paper and will be the subject of a future study.

**2.** Let us construct the approximating extremal control $u_\delta^\circ$. We begin with some remarks about notation. The subscript $\delta$ emphasizes that the realizations $u_\delta[t]$ of the control $u_\delta$ are constant in the interval $\tau_k \leqslant t < \tau_{k+1}$ ($\tau_k = t_0 + k\delta$, $\delta > 0$, $k = 0, 1, \ldots$). With the chosen controls $u$ and $v$ symbol $T^\varepsilon_{u,v}$ denotes the instant when the inequality

$$\|\{y[T^\varepsilon_{u,v}] - z[T^\varepsilon_{u,v}]\}_m\| \leqslant \varepsilon \tag{2.1}$$

holds for the first time for the motions $y[t]$ and $z[t]$ under consideration.

(The symbol $\|w\|$ denotes the Euclidean norm of the vector $w$.) By definition, the control $u_\delta$ defined for all sufficiently small $\delta > 0$ ensures convergence of the motions $y[t]$ and $z[t]$ not later than at the instant $t = T^\circ$ provided that the condition [10]

$$\sup_{\varepsilon > 0}[\lim_{\delta \to 0}\sup[\sup_v T^\varepsilon_{u,v}]] \leqslant T^\circ \tag{2.2}$$

is fulfilled.

The symbols $G^{(1)}[y, \eta]$ and $G^{(2)}[z, \eta]$ denote the attainability domains ([7], p. 116) for the motions $y$ and $z$, respectively (from the states $y[t] = y$, $z[t] = z$ by the instant $\vartheta = t + \eta$). The symbols $G^{(1)}[y, \eta; \varepsilon]$ and $G^{(2)}[z, \eta; \varepsilon]$ represent the closed $\varepsilon$-neighborhoods of the domains $G^{(1)}$ and $G^{(2)}$ (in a metric defined by some norm $\gamma$).

The extremal control

$$u_\delta^\circ[t] = u_\delta^\circ[y[\tau_k], \ z[\tau_k], \ \eta[\tau_k]] \tag{2.3}$$
$$(\tau_k \leqslant t < \tau_{k+1})$$

is formed as follows. The quantities $y[t]$, $z[t]$ vary in accordance with Eqs. (1.1), (1.2), where $v = v[t]$ and $u = u_\delta^\circ[t]$. The quantities $\eta[\tau_k]$ are defined recurrently. For $\tau_0 = t_0$ we set $\eta[\tau_0] = \vartheta^\circ[\tau_0] - \tau_0$, where $\vartheta = \vartheta^\circ[\tau_0]$ is the instant of absorption of the process $z$ by the process $y$, when $G^{(2)}[z[\tau_0], \vartheta - \tau_0] \subset G^{(1)}[y[\tau_0], \vartheta - \tau_0]$ for the first time. Now let the quantities $y[\tau_k]$, $z[\tau_k]$ be realized at the instant $t = \tau_k$, and let us determine the instant of absorption $\vartheta^\circ[\tau_k]$ for these quantities. If $\vartheta^\circ[\tau_k] \leqslant \tau_{k-1} + \eta[\tau_{k-1}]$, then we set $\eta[\tau_k] = \vartheta^\circ[\tau_k] - \tau_k$; but if $\vartheta^\circ[\tau_k] > \tau_{k-1} + \eta[\tau_{k-1}]$, then $\eta[\tau_k] = \eta[\tau_{k-1}] - \delta$.

Let us choose some function $\xi(\delta)$ satisfying the condition

$$\lim \xi(\delta) = 0 \quad \text{as} \quad \delta \to 0 \quad (\xi \geqslant \delta) \tag{2.4}$$

For known $y[\tau_k]$, $z[\tau_k]$, $\eta[\tau_k]$, among the permissible program controls $u(\tau)$ ($\tau_k \leqslant \tau < \tau_k + \xi$) restricted by the condition $u(\tau) \in U$ there exists a control $u_*(\tau)$ which delivers the minimum $\varepsilon^*[\tau_k]$ of the quantity $\varepsilon$ satisfying the condition

$$G^{(2)}[z[\tau_k], \ \eta[\tau_k]] \subset G^{(1)}[y_u(\tau_k + \xi), \ \eta[\tau_k]; \ \varepsilon] \tag{2.5}$$

Here $y_u(\tau_k + \xi)$ is the state to which the system

$$\frac{dy}{d\tau} = f^{(1)}[y, u] \tag{2.6}$$

is brought by the control $u(\tau)$ by the instant $\tau = \tau_k + \xi$ (from the state $y[\tau_k]$). The control $u_\delta^\circ$ (2.3) is now defined by

$$u_\delta^\circ[t] = \frac{1}{\delta}\int_{\tau_k}^{\tau_{k+1}} u_*(\tau)\,d\tau \tag{2.7}$$

**3.** Now let us formulate a definition of the property of stable absorption of the process $z$ by the process $y$ for the system (1.1)–(1.3).

Let $0 < \eta_0 \leqslant \eta \leqslant \eta^\circ < \infty$, $\delta > 0$, and let some values of $z[t]$, $y[t]$ such that

$$G^{(2)}[z[t], \ \eta] \subset G^{(1)}[y[t], \ \eta; \ \varepsilon_*] \quad (\varepsilon_* \geqslant \varepsilon_0 > 0) \tag{3.1}$$

be realized.

Further, let some value $z[t+\delta]$ be realized. We say that the process $z$ is "stably" absorbed by the process $y$ if, whatever the possible realizations $z[t]$, $z[t+\delta]$, $y[t]$ and whatever the quantities $\eta_0$, $\eta^\circ$, $\varepsilon_0$ for all sufficiently small $\delta>0$ there exists among the permissible controls $u(\tau)$ $(t\leqslant\tau<t+\delta)$ a control $u^*(\tau)$ which brings the system (2.6) to a state $y_u^*(t+\delta)$ satisfying the condition

$$G^{(2)}[z[t+\delta],\ \eta-\delta]\subset G^{(1)}[y_u^*(t+\delta),\ \eta-\delta;\ \varepsilon^*] \qquad (3.2)$$

where

$$\varepsilon^*\leqslant(1+\beta(\eta_0\eta^\circ)\delta)\,\varepsilon_*\ (\beta<\infty) \qquad (3.3)$$

The following statement is valid.

Theorem 3.1. If the process $z$ is stably absorbed by the process $y$, then the extremal control $u_\delta^\circ$ (2.7) ensures convergence of the motions $y[t]$ and $z[t]$ not later than at the instant $t=\vartheta^\circ[t_0]$. (We assume, of course, that an instant of absorption $\vartheta^\circ[t_0]<\infty$ exists for the given $y[t_0]$ and $z[t_0]$).

To prove the theorem we need merely show that the quantity $\varepsilon^*\cdot[\tau_k]$ which minimizes $\varepsilon$ in condition (2.5) remains smaller than any number $\varepsilon^\circ>0$ chosen in advance, provided that the scheme interval $\delta>0$ is sufficiently small. To this end we estimate the changes in the given quantity with the time $\tau_k$. Clearly, we need only consider the case where $\varepsilon^*[t]\geqslant\varepsilon_0>0$ and $\eta_0\leqslant\eta\leqslant\eta^\circ$. Thus, let us estimate the change in the quantity $\varepsilon^*[t]$ in a single step $[\tau_k,\tau_{k+1})$. If the control $u_*(t)$ has operated over the given interval $\tau_k\leqslant t<\tau_{k+1}$, then condition (3.3) yields the inequality

$$\varepsilon^*(\tau_{k+1})\leqslant(1+\beta(\eta_0,\ \eta^\circ)\delta)\,\varepsilon^*[\tau_k] \qquad (3.4)$$

However, the control which actually operates during the interval $\tau_k\leqslant t<\tau_{k+1}$ is the average control $u_\delta^\circ[t]$ (2.7). This averaging in our estimate of the quantity $\varepsilon^*[\tau_{k+1}]$ yields an effect of a higher order of smallness in $\delta$. In this fashion we arrive at the estimate

$$\varepsilon^*[\tau_{k+1}]\leqslant(1+\beta(\eta_0,\ \eta^\circ)\delta)\,\varepsilon^*[\tau_k]+o(\delta) \qquad (3.5)$$

which implies the validity of the theorem.

Note 3.1. Let the attainability domains $G^{(i)}$ be closed. This applies in a broad class of cases. Further, let the domain $G^{(1)}[y,\eta]$ be convex. It then constitutes the intersection of its support half-spaces ([11], p. 781) and is described by the relation

$$\rho^{(1)}[l,y,\eta]-l'q\geqslant0$$

which every point $q\in G^{(1)}$ must satisfy for all possible values of the vector $l$. (The prime indicates transposition). Let $\dot G^{(2)}[z,\eta]$ be the convex shell of the domain $G^{(2)}[z,\eta]$ and let the domain $\dot G^{(2)}[z,\eta]$ be described by

$$\rho^{(2)}[l,z,\eta]-l'q\geqslant0$$

The property of stable absorption of the process $z$ by the process $y$ is then fulfilled if for small $\delta>0$ we have

$$\max_{u(\tau)}(\min_l[\rho^{(1)}(l,y_u(t+\delta),\eta-\delta)]-\rho^{(2)}[l,z[t+\delta],\eta-\delta]]+(1+\beta\delta)\,\varepsilon_*)\geqslant0 \quad (3.6)$$

provided that

$$\min_l[\rho^{(1)}[l,y[t],\eta]-\rho^{(2)}[l,z[t],\eta]]+\varepsilon_*\geqslant0$$

when $\gamma^*[l]=1$. Here $\gamma^*[l]$ is the norm of the vector $l$ in the appropriate metric. When the function in the left side of (3.6) is convex in $l$ and concave in $u$, and when the set $U$ in condition (1.3) is convex, the operations $\max_u$ and $\min_l$ in (3.6) can be permuted, with the result that the determination of the function $u_*(\tau)$ now rests on the ordinary conditions of the maximum principle [12]. Such ordinary conditions are closely related to the cases of regular absorption of the process $z$ by the process $y$ considered in [9 and 10]. Also of interest, however, is the special case in which the absorption of the process $z$ by the process $y$ is irregular, and when the functions under consideration are not convex in $l$. Computation of the function $u_*(\tau)$ then entails additional complications.

Note 3.2. The limiting motions $y^\circ[t]$ generated by the approximating scheme for the control $u_\delta^\circ$ as $\delta \to 0$ can be formalized within the framework of the general solutions [13] obtained by means of discontinuous differential equations. The control $u^\circ$ is constructed formally in the following way. If the inclusion

$$G^{(2)}[z[t], \eta[t]] \subset G^{(1)}[y_u^*(t+\xi), \eta[t]]$$

is valid for given $y[t], z[t], \eta[t], \xi[t]$, then the function $u^\circ(y, z, \eta, \xi)$ is assumed to be non-singlevalued at such a point $\{y, z, \eta, \xi\}$, and can assume any values which satisfy the prescribed restriction $u^\circ \in U$. However, if only the inclusion

$$G^{(2)}[z[t], \eta[t]] \subset G^{(1)}[y_u^*(t+\xi), \eta[t]; \varepsilon^*] \tag{3.7}$$

is fulfilled for the given $y, z, \eta, \xi$, then $u^\circ[t]$ is given by

$$u^\circ[t] = u^\circ(y[t], z[t], \eta[t], \xi[t]) = \frac{1}{\xi[t]} \int_t^{t+\xi} u_*(\tau)\, d\tau \tag{3.8}$$

Some remarks concerning the character of variation of the variables $\eta[t]$ and $\xi[t]$. are called for here. We assume that the function $\eta[t]$ is described by the differential equation $d\eta / dt = -1$, and that $\xi[t]$ is a continuous nondecreasing function restricted over its intervals of increase by the condition

$$\xi[t] = \alpha(t, \eta[t]) \sup_{t_* \leqslant \tau \leqslant t} \varepsilon^*[\tau] \tag{3.9}$$

where $\alpha(t, \eta)$ is a function satisfying the appropriate conditions of smallness. Under these assumptions the solution $y[t]$ of Eq. (1.1) is defined as an absolutely continuous function which satisfies Eq. (1.1) for $u = u^\circ[t]$ for almost all values of $t$.

We can verify the existence of the solution $y[t]$ by taking the limit of the approximating solutions $y_\delta[t]$ constructed according to a scheme similar to that constructed in Sect. 2

The conditions under which the required solutions $y^\circ[t]$ exist reduce here to the continuity of the function $u^\circ$ in the range where it is single-valued, and to certain known functional restrictions on $f^{(1)}[y, u]$ which are typical for problems on the construction of generalized solutions of discontinuous equations [13]. In any case, these conditions are fulfilled for a broad class of linear systems under convex restrictions on the control $u$. The condition of stable absorption of the process $z$ by the process $y$ then proves that for motions $y^\circ[t], z[t], \eta[t], \xi[t]$ satisfying the initial conditions $\eta[t_0] = \vartheta[t_0] - t_0$, $\xi[t_0] = 0$ the extremal control $u^\circ$ ensures encounter not later than at the instant $t = \vartheta^\circ[t_0]$. The validity of this statement follows from the fact that the function $\varepsilon^*[t]$ does not increase during the generalized motions $y^\circ[t]$ under consideration and therefore remains equal to zero all the time. This fact follows in turn from the circumstance that for $\varepsilon^*[t] \neq 0$ the quantity $\varepsilon^*[t]$ cannot increase too rapidly for $t \geqslant t_*$, since for $u = u^\circ$ the function $\varepsilon^*[t]$ satisfies estimates similar to estimate (3.5) which is valid for a discrete scheme. We note that the motion $y^\circ[t]$ is generally realized in the form of slip conditions. We also note that the pursuit process is improved for the pursuer if we replace Eq. $d\eta / dt = -1$ for the function $\eta[t]$ by the differential inequality

$d\eta / dt \leqslant -1$ and imposing certain other restrictions on the function $\eta[t]$. However, this complicates determination of the generalized solution $y^*[t]$ of Eq. (1.1).

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# STABILITY OF MOTION OVER A FINITE TIME INTERVAL

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A family of necessary and sufficient conditions for the stability and instability of motion over a finite time interval is constructed. This is made possible by a generalization of Kamenkov's formulation of the problem of stability over a finite time interval.

**1.** In his investigation of mechanical systems whose perturbed motion is described by the equations

$$\frac{dx_i}{dt} = X_i(t;\ x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{1.1}$$

where $X_i$ are real functions of real variables which vanish for $x_i = 0$ $(i = 1, \ldots, n)$ and can be expanded in series in whole nonnegative powers of $x_i$ in the neighborhood of the origin ($x_i = 0$), Kamenkov introduced the following definition of stability of motion